... and now for something completely different…

Set Theory

Actually, you will see that logic and set theory are very closely related.

Set Theory

- Set: Collection of objects (called elements)
- $a \in A$ "a is an element of A" "a is a member of A"
- $a \notin A$ "a is not an element of A"
- $A = \{a_1, a_2, \ldots, a_n\}$ "A contains $a_1, \ldots, a_n$"
- Order of elements is insignificant
- It does not matter how often the same element is listed (repetition doesn't count).

Set Equality

Sets A and B are equal if and only if they contain exactly the same elements.

Examples:

- $A = \{9, 2, 7, -3\}$, $B = \{7, 9, -3, 2\}$: $A = B$
- $A = \{dog, cat, horse\}$,
  $B = \{cat, horse, squirrel, dog\}$: $A \neq B$
- $A = \{dog, cat, horse\}$,
  $B = \{cat, horse, dog, dog\}$: $A = B$

Examples for Sets

"Standard" Sets:

- Natural numbers $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$
- Integers $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Positive Integers $\mathbf{Z}^+ = \{1, 2, 3, 4, \ldots\}$
- Real Numbers $\mathbf{R} = \{47.3, -12, \pi, \ldots\}$
- Rational Numbers $\mathbf{Q} = \{1.5, 2.6, -3.8, 15, \ldots\}$

(correct definitions will follow)

Examples for Sets

- $A = \emptyset$ "empty set/null set"
- $A = \{z\}$ Note: $z \in A$, but $z \neq \{z\}$
- $A = \{\{b, c\}, \{c, x, d\}\}$ set of sets
- $A = \{\{x, y\}\}$ Note: $\{x, y\} \in A$, but $\{x, y\} \neq \{\{x, y\}\}$
- $A = \{x \mid P(x)\}$ "set of all x such that P(x)"

  P(x) is the membership function of set A

  $\forall x\, (P(x) \rightarrow x \in A)$
- $A = \{x \mid x \in \mathbf{N} \wedge x > 7\} = \{8, 9, 10, \ldots\}$
  "set builder notation"

Examples for Sets

We are now able to define the set of rational numbers Q:

$\mathbf{Q} = \{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z}^+\}$, or

$\mathbf{Q} = \{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z} \wedge b \neq 0\}$

And how about the set of real numbers R?

$\mathbf{R} = \{r \mid r \text{ is a real number}\}$

That is the best we can do. It can neither be defined by enumeration nor builder function.

Subsets

$A \subseteq B$ "A is a subset of B"

$A \subseteq B$ if and only if every element of A is also an element of B.

We can completely formalize this:

$A \subseteq B \Leftrightarrow \forall x\ (x \in A \rightarrow x \in B)$

Examples:

$A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? true

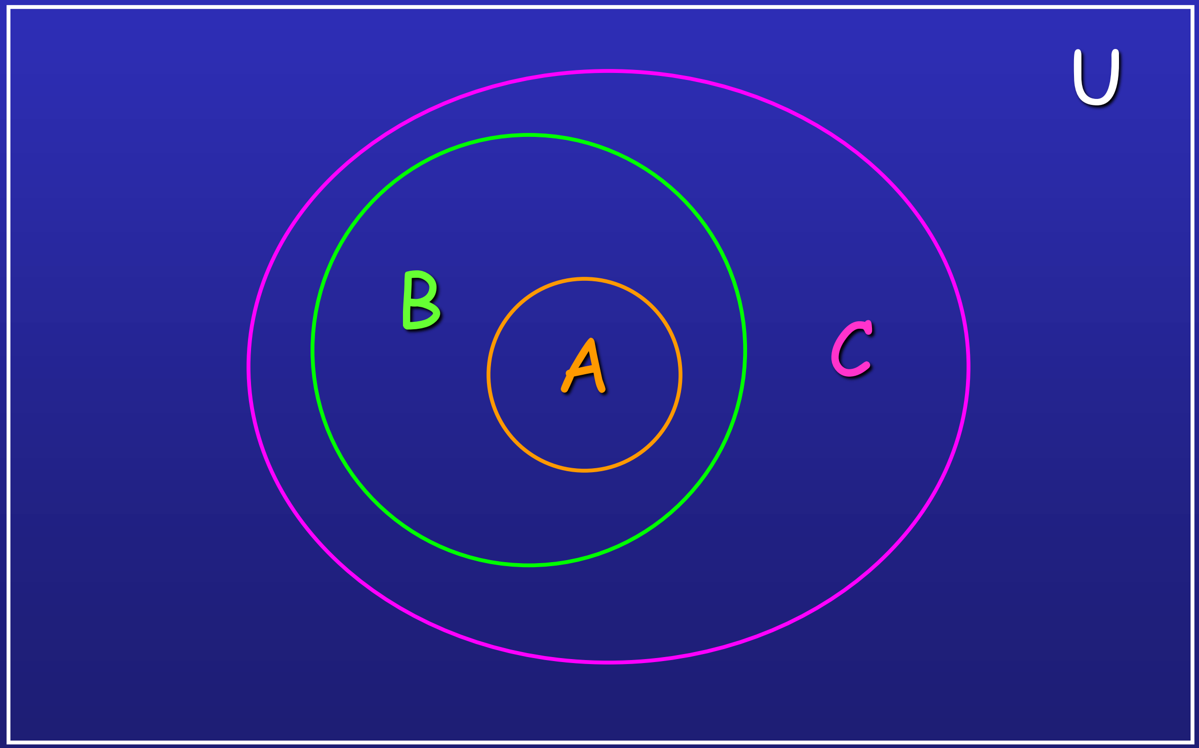
$A = \{3, 3, 3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? true

$A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $A \subseteq B$? false

Subsets

Useful rules:

- $A = B \Leftrightarrow (A \subseteq B) \land (B \subseteq A)$
- $(A \subseteq B) \land (B \subseteq C) \Rightarrow A \subseteq C$ (see Venn Diagram)



Subsets

Useful rules:

- $\emptyset \subseteq A$ for any set A
  (but $\emptyset \in A$ may not hold for any set A)
- $A \subseteq A$ for any set A

Proper subsets:

$A \subset B$ "A is a proper subset of B"

$A \subset B \Leftrightarrow \forall x\ (x \in A \rightarrow x \in B) \wedge \exists x\ (x \in B \wedge x \notin A)$

or

$A \subset B \Leftrightarrow \forall x\ (x \in A \rightarrow x \in B) \wedge \neg \forall x\ (x \in B \rightarrow x \in A)$

Cardinality of Sets

If a set S contains n distinct elements, $n \in \mathbf{N}$, we call S a finite set with cardinality n.

Examples:

$A = \{\text{Mercedes, BMW, Porsche}\}$, $|A| = 3$

$B = \{1, \{2, 3\}, \{4, 5\}, 6\}$ $|B| = 4$

$C = \emptyset$ $|C| = 0$

$D = \{ x \in \mathbf{N} \mid x \leq 7000 \}$ $|D| = 7001$

$E = \{ x \in \mathbf{N} \mid x \geq 7000 \}$ E is infinite!

The Power Set

$P(A)$ "power set of A" (also written as 2^A)

$P(A) = \{B \mid B \subseteq A\}$ (contains all subsets of A)

Examples:

$A = \{x, y, z\}$

$P(A) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$

$A = \emptyset$

$P(A) = \{\emptyset\}$

Note: $|A| = 0$, $|P(A)| = 1$

The Power Set

Cardinality of power sets: $| P(A) | = 2^{|A|}$

- Imagine each element in A has an "on/off" switch
- Each possible switch configuration in A corresponds to one subset of A, thus one element in P(A)

A	1	2	3	4	5	6	7	8
x	x	x	x	x	x	x	x	x
y	y	y	y	y	y	y	y	y
z	z	z	z	z	z	z	z	z

- For 3 elements in A, there are $2 \times 2 \times 2 = 8$ elements in P(A)

Cartesian Product

The ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ is an ordered collection of n objects.

Two ordered n-tuples $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are equal if and only if they contain exactly the same elements in the same order, i.e. $a_i = b_i$ for $1 \leq i \leq n$.

The Cartesian product of two sets is defined as:

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Cartesian Product

Example:

A = {good, bad}, B = {student, prof}

$A \times B$ = {(good, student), (good, prof), (bad, student), (bad, prof)}

$B \times A$ = {(student, good), (prof, good), (student, bad), (prof, bad)}

Example: A = {x, y}, B = {a, b, c}

$A \times B$ = {(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)}

Cartesian Product

Note that:

- $A \times \emptyset = \emptyset$
- $\emptyset \times A = \emptyset$
- For non-empty sets A and B: $A \neq B \Leftrightarrow A \times B \neq B \times A$
- $|A \times B| = |A| \cdot |B|$

The Cartesian product of two or more sets is defined as:

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}$

Set Operations

Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cup B = \{a, b, c, d\}$

Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cap B = \{b\}$

Cardinality: $|A \cup B| = |A| + |B| - |A \cap B|$